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(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIRST SEMESTER EXAMINATION, DECEMBER 2011

FIRST YEAR

MATHEMATICS (Honours)

Paper : I

Date : 16/12/2011

Time : 11am – 3 pm

Full Marks : 100

[Use separate answer-books for each group]

Group–A

[Throughout, $\mathbb{R}$ and $\mathbb{N}$ respectively denote the set of all real numbers and natural numbers]

1. Answer **any five** questions: 5x5

a) i) For three subsets *A*, *B*, *C* of a set *S*, if $(A \cap C) \cup (B \cap C') = \phi$, prove that $A \cap B = \phi$ where C' is the complement of C in *S*.

ii) Let $f : X \to Y$ be a function and let $f(A \cap B) = f(A) \cap f(B)$ for all non-empty subsets *A* and *B* of *X*. Prove that *f* is injective. 3+2

b) i) Define a bijective map from [0,1] onto $[0,1] \cup \{\sqrt{2}, \sqrt{3}\}$.
Give justifications.

ii) Find the order of the permutation $(1234)(56) \in S_6$. 3+2

c) i) Define ρ on $\mathbb{R} - \{0\}$ by '$x \rho y$ iff $\frac{x}{y} > 0;\ x, y \in \mathbb{R} - \{0\}$'. Show that ρ is an equivalence relation. Find all equivalence classes.

ii) Define an equivalence relation on {1,2,3}. (2+2)+1

d) Let $G = \{(a,b) : a, b \in \mathbb{R}, a \neq 0\}$. Define a binary operation '$\circ$' on *G* by $(a,b) \circ (c,d) = (ac, b+d)$ for all $(a,b),(c,d) \in G$. Show that $(G, \circ)$ is a group. Also prove that *G* has no element of order 3. 4+1

e) Prove that the intersection of any family of subgroups of a group *G* is a subgroup of *G*. Give an example to show that the result may not hold in case of union. 3+2

f) Prove that every subgroup of a cyclic group is cyclic. 5

g) Consider the group $\mathbb{R} \times \mathbb{R} = \{(a,b) : a, b \in \mathbb{R}\}$ under componentwise addition of real numbers. Let $H = \{(x, 3x) \in \mathbb{R} \times \mathbb{R}\}$. Show that *H* is a subgroup of $\mathbb{R} \times \mathbb{R}$ and then show that *H* represents geometrically all points on the line *y* = 3*x*. Show that all the points of the coset (2,5) + *H* are on the straight line *y* = 3*x* – 1. 2+1+2

h) State and prove Lagrange's theorem on finite groups. 5

2. Answer **any five** questions: 5x5

a) i) State Archimedean property of real numbers and use it to show that $\lim_{n \to \infty} \frac{1}{n} = 0$. 3

ii) Let *S* and *T* be two non-empty bounded subsets of $\mathbb{R}$ and $U = \{x + y; x \in S, y \in T\}$. Prove that $\inf U = \inf S + \inf T$. 2

b) i) Let $S=(0,1]$ and $T=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$. Show that $S-T$ is an open set. 2

ii) Prove that the interior Int S of a set S is an open set. 3

c) i) Determine whether the set $S=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}\bigcup\left\{1+\frac{1}{n}:n\in\mathbb{N}\right\}$ is closed. 2

ii) Prove that the derived set of a set $S(\subset\mathbb{R})$ is a closed set. 3

d) i) Prove that the complement of an open set in $\mathbb{R}$ is a closed set.

ii) Show that the union of a finite number of closed sets in $\mathbb{R}$ is a closed set. 5

e) i) Define an uncountable set. 1

ii) Show that the interval (0,1) is uncountable. 4

f) i) Let the sequence $\{x_n\}$ be defined inductively by $x_1=1$, $x_{n+1}=\frac{1}{4}(2x_n+3)$ for $n\geq 1$. Show that $\{x_n\}$ converges and find the limit. 3

ii) Prove that every Cauchy sequence is bounded. 2

g) Define a subsequence of the sequence $\{x_n\}$ of real numbers. Prove that every bounded sequence of real numbers has a convergent subsequence. 1+4

h) i) Prove that if a sequence $\{x_n\}$ is convergent then $\overline{\lim}\,x_n=\underline{\lim}\,x_n$. 2

ii) Discuss existence of $\lim\limits_{x\to 0}\left(\cos\frac{1}{x}\right)$. 3

Group–B

3. Answer **any five** questions : 5×5 = 25

a) Show that the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is right angled if $(a+b)(al^2+2hlm+bm^2)=0$. 5

b) A circle is drawn through the focus of the parabola $\frac{2a}{r}=1+\cos\theta$ to touch it at the point $\theta=\alpha$. Show that its equation is $r\cos^3\frac{\alpha}{2}=a\cos\left(\theta-\frac{3\alpha}{2}\right)$. 5

c) Reduce the equation $3x^2+10xy+3y^2-2x-14y-13=0$ to its canonical form. Find its nature and length of the latus rectum. 5

d) Show that the pole of any tangent to the hyperbola $xy=c^2$ with respect to the circle $x^2+y^2=a^2$ lies on concentric and similar hyperbola. 5

e) The normal at a variable point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts the diameter CD conjugate to CP at Q. Find the locus of Q. 5

f) Prove the following by vector method:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

symbols are of usual meaning. 5

g) If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors then express $\vec{b}\times\vec{c},\ \vec{c}\times\vec{a},\ \vec{a}\times\vec{b}$ in terms of $\vec{a}, \vec{b}, \vec{c}$. 5

h) Solve for $\vec{r}$, $K\vec{r}+\vec{r}\times\vec{a}=\vec{b}$ where K is a given nonzero scalar and $\vec{a}, \vec{b}$ are two given vectors. 5

4. Answer **__any five__** questions: 5x5 = 25

a) Obtain the differential equation of all circles each of which touches the axis of x at the origin. 5

b) Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, λ is the parameter. 5

c) Reduce the differential equation $xp^2-2yp+x+2y=0$ $\left(p=\frac{dy}{dx}\right)$ to Clairaut's form by transforming $x^2=u$ and $y-x=v$. Find the general solution and singular solution (if it exists). 1+3+1

d) Solve: $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\log y$. 5

e) Solve, by the method of variation of parameters, the equation $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$. 5

f) Solve by method of undetermined coefficients $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$. 5

g) Solve; $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$. 5

h) Show that $\sin x\frac{d^2y}{dx^2}-\cos x\frac{dy}{dx}+2y\sin x=0$ is exact and solve it completely. 5